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Orthonormal coherent states on von Neumann lattices

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Abstract

An averaging procedure in phase plane is developed leading to orthonormality of coherent states on a von Neumann lattice. These states correspond to entire unit cells of area h (Planck constant) in the phase plane and they can be specified by $(mb, n\,\frac{2\pi}{b}\,\hbar)$, where m and n are integers and b is a constant related to the spread of the harmonic oscillator ground state. The product of uncertainties for the co-ordinate x and momentum p in these states is very close to $\hbar$. This is by a factor of 2π smaller than the unit cell area, and makes it, in principle, possible to measure simultaneously x and p for the location of a unit cell in the phase plane.

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Since their discovery by Schrödinger [1], coherent states have been widely used in many fields of physics [2–5]. A discrete subset of coherent states was introduced by von Neumann [6], by assigning a single state to a unit cell of area h (Planck constant) in the phase plane [7]. Being coherent states the von Neumann states are non-orthogonal and attempts to make them orthogonal have led to the Balian–Low theorem [8–10] according to which one can modify the coherent states in the von Neumann set in order to make them orthogonal but for the modified states the product $\Delta x\,\Delta p$ of the uncertainties diverges, which actually means that they completely lose their classical nature. In building his set von Neumann has chosen a single coherent state in each unit cell. Such a state is labelled by $(mb, n\,\frac{2\pi}{b}\,\hbar)$, for the x and p co-ordinates in the phase plane and with b being an arbitrary constant. This is also the label for the unit cell. But in each unit cell there is an infinite number of coherent states, which means that there is much freedom in building different von Neumann sets. In this letter we use this freedom and we develop an averaging procedure for constructing states that are related to an entire $(mb, n\,\frac{2\pi}{b}\,\hbar)$-cell in the phase plane. This procedure leads to very striking and previously unachievable results. The most important among them being the orthonormality of the coherent states when averaged over a unit cell of area h in the phase plane. We use this averaging procedure for calculating the matrix elements of x and p and for their uncertainties Δx and Δp in these states. It turns out that the expectation values $\overline{x}$ and $\overline{p}$ are correspondingly

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$\overline{x} = mb$ and $\overline{p} = n\,\frac{2\pi}{b}\,\hbar$ in these states, while $\Delta x\,\Delta p$ (the product of uncertainties) for them is very close to $\hbar$, meaning that they preserve their classical nature. Having established the orthonormality of coherent states on a von Neumann lattice, we also define a probability distribution $|\langle m, n|\psi\rangle|^2$ for any state vector $|\psi\rangle$. This distribution is obtained by the above averaging procedure over a unit cell of the Husimi function [11, 12].

When dealing with coherent states the shift operator

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a) = \exp\left[\frac{\mathrm{i}}{\hbar}\,(x\overline{p} - p\overline{x})\right] \tag{1}$$

is of much importance. Here a is the annihilation operator (a^+ is the creation operator which is a Hermitian conjugate to a), whose eigenstates $|\alpha\rangle$ are the coherent states

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{2}$$

where

$$a = \frac{1}{\lambda\sqrt{2}}\left(x + \mathrm{i}\,\frac{\lambda^2}{\hbar}\,p\right) \qquad \alpha = \frac{1}{\lambda\sqrt{2}}\left(\overline{x} + \mathrm{i}\,\frac{\lambda^2}{\hbar}\,\overline{p}\right). \tag{3}$$

λ in equation (3) is a constant which generally appears in the harmonic oscillator states ($\lambda^2 = \frac{\hbar}{m\omega}$). A convenient way to write a coherent state is

$$D(\alpha)|0\rangle = |\alpha\rangle \tag{4}$$

where $D(\alpha)$ is the shift operator in equation (1) and $|0\rangle$ is the ground state of a harmonic oscillator. The coherent states are known to be non-orthogonal

$$|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2) \tag{5}$$

and highly overcomplete [3]. The von Neumann discrete subset of coherent states

$$D(\alpha_{mn})|0\rangle = |\alpha_{mn}\rangle \tag{6}$$

with

$$\alpha_{mn} = \frac{1}{\lambda\sqrt{2}}\left(ma + \mathrm{i}\,\frac{2\pi}{a}\,\lambda^2 n\right) \qquad m, n = 0, \pm 1, \pm 2, \ldots \tag{7}$$

on the von Neumann lattice in the phase plane, is still complete (actually overcomplete by one state [5, 13, 14]), but clearly also non-orthogonal for $\alpha_{mn} \neq \alpha_{m'n'}$. One can build infinitely many non-equivalent von Neumann sets in the following way:

$$D(\alpha_{mn})\,D(\beta)|0\rangle \qquad \text{and the ‘bra’:} \quad \langle 0|D(-\beta)\,D(-\alpha_{mn}) \tag{8}$$

where

$$\beta = \frac{1}{\lambda\sqrt{2}}\left(\overline{X} + \frac{\mathrm{i}\,\lambda^2}{\hbar}\,\overline{P}\right) \qquad \text{and} \qquad -\frac{b}{2} \leqslant \overline{X} \leqslant \frac{b}{2} \quad -\frac{\pi}{b}\,\hbar \leqslant \overline{P} \leqslant \frac{\pi}{b}\,\hbar. \tag{9}$$

Here $\overline{X}$ and $\overline{P}$ are confined to the zero unit cell of the von Neumann lattice. We now use this freedom to prove orthonormality of the von Neumann set when averaged over a unit cell in the phase plane. Namely,

$$\frac{1}{h}\iint \langle 0|\,D(-\beta)\,D(-\alpha_{mn})|D(\alpha_{m'n'})\,D(\beta)\,|0\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} = \delta_{mm'}\,\delta_{nn'}. \tag{10}$$

Equation (10) is easiest to prove using the kq-representation [15], in which the ground state of a harmonic oscillator is

$$\langle kq|0\rangle = \left(\frac{b}{2\,\lambda\,\pi^{3/2}}\right)^{1/2} \exp\left(-\frac{q^2}{2\,\lambda^2}\right)\,\theta_3(z|\tau) \tag{11}$$

with $z = \frac{kb}{2} - \frac{\mathrm{i}}{2}\frac{bq}{\lambda^2}$, $\tau = \mathrm{i}\,\frac{b^2}{2\pi\lambda^2}$. In equation (11) $\theta_3(z|\tau)$ is a Jacobi theta function [16]. The convenience of using the kq-representation stems from the fact that in it the von Neumann set in equation (6) assumes the simple form [7]

$$D(\alpha_{mn})\,\langle kq|0\rangle = (-1)^{mn}\exp\left(-\mathrm{i}kbm + \mathrm{i}q\,\frac{2\pi}{b}\,n\right)\langle kq|0\rangle. \tag{12}$$

In the kq-representation, the left-hand side of equation (10) becomes

$$\frac{(-1)^{mn+m'n'}}{h}\int \exp\left[-\mathrm{i}kb(m'-m) + \mathrm{i}q\,\frac{2\pi}{b}\,(n'-n)\right]|D(\beta)\langle kq|0\rangle|^2\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P}\,\mathrm{d}k\,\mathrm{d}q. \tag{13}$$

The application of the shift operator $D(\beta)$ (equation (1)) to any kq-function will shift k to $k - \frac{\overline{P}}{\hbar}$ and q to $q - X$ (a phase will also be added). In equation (13) the function $|\langle k - \frac{\overline{P}}{\hbar}, q - \overline{X}|0\rangle|^2$ is periodic in all four arguments. This means that from the normalization of $\langle kq|0\rangle$ it follows that

$$\iint \left|\left\langle k - \frac{\overline{P}}{\hbar}, q - \overline{X}\right|0\right\rangle\right|^2 \mathrm{d}\overline{X}\,\mathrm{d}\overline{P} = \hbar. \tag{14}$$

The remaining integration over k and q in equation (13) completes the proof of equation (10). It should be pointed out that in no place was the fact used that the starting state was $\langle kq|0\rangle$. This means that no matter with what normalized square integrable state $\langle kq|\psi\rangle$ one starts, the von Neumann set (equation (12)) will be orthonormal when averaged over a unit cell in the phase plane. This demonstrates the universality of the averaging procedure over a unit cell of area h in the phase plane.

Having determined in what sense the coherent states are orthonormal, we now use the same averaging procedure for operators. In what follows we show how to calculate the matrices on a von Neumann lattice for the fundamental operators x and p. A straightforward result is

$$\begin{aligned}&\langle 0|D(-\beta)\,D(-\alpha_{mn})|\,a\,|D(\alpha_{m'n'})\,D(\beta)|0\rangle\\ &\qquad = (\alpha_{m'n} + \beta)\langle 0|D(-\beta)\,D(-\alpha_{mn})|D(\alpha_{m'n'})\,D(\beta)|0\rangle\end{aligned} \tag{15}$$

and a similar result for the creation operator a^+. Using the expressions for x and p by means of a and a^+ and by averaging over the unit cell we find

$$\frac{1}{h}\iint \langle 0|D(-\beta)\,D(-\alpha_{mn})|\,x\,|D(\alpha_{m'n'})\,D(\beta)|0\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} = ma\,\delta_{mm'}\,\delta_{nn'} + \overline{X}_{mn,m'n'} \tag{16}$$

$$\frac{1}{h}\iint \langle 0|D(-\beta)\,D(-\alpha_{mn})|\,p\,|D(\alpha_{m'n'})\,D(\beta)|0\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} = n\,\frac{2\pi}{a}\,\hbar\,\delta_{mm'}\,\delta_{nn'} + \overline{P}_{mn,m'n'} \tag{17}$$

where

$$\overline{X}_{mn,m'n'} = \frac{1}{h}\iint \overline{X}\langle 0|D(-\beta)\,D(-\alpha_{mn})|\,D(\alpha_{m'n'})\,D(\beta)|0\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} \tag{18}$$

$$\overline{P}_{mn,m'n'} = \frac{1}{h}\iint \overline{P}\langle 0|D(-\beta)\,D(-\alpha_{mn})|\,D(\alpha_{m'n'})\,D(\beta)|\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P}. \tag{19}$$

Assuming, by definition, that the integration over $\overline{X}$ is from $-\frac{b}{2}$ to $\frac{b}{2}$ and over $\overline{P}$ from $-\frac{\pi}{b}\,\hbar$ to $\frac{\pi}{b}\,\hbar$, we immediately see that $\overline{X}_{mn,m'n'}$ and $\overline{P}_{mn,m'n'}$ have no diagonal elements. Our averaging procedure therefore gives results for the expectation values $\overline{x}$ and $\overline{p}$ of the co-ordinate and the momentum

$$\overline{x} = mb \qquad \overline{p} = n\,\frac{2\pi}{b}\,\hbar \tag{20}$$

that were anticipated by von Neumann when he first introduced his discrete set of coherent states almost 70 years ago [6]. The discrete values of the x and p spectra on the phase plane in equation (20) express their commutative parts, and they label the different cells. As we are going to show now, the non-diagonal elements in equations (18) and (19) can be easily calculated; they are small and they lead to an uncertainty product $\Delta x\, \Delta p$ of the order of $\hbar$.

For finding $\overline{X}_{mn,m'n'}$, $\overline{P}_{mn,m'n'}$ we have to calculate an integral of the form (see equation (13))

$$I = \frac{1}{h}\iint L\,|D(\beta)\langle kq|0\rangle|^2\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} \tag{21}$$

where L can be either $\overline{X}$ or $\overline{P}$. When $L = \overline{X}$, then using the expression in equation (11), the integration over $\overline{P}$ becomes trivial, and we get for I in equation (21) the expression

$$I = \frac{1}{2\pi}\left(\frac{1}{\pi\,\lambda^2}\right)^{1/2}\int_{-b/2}^{b/2}\overline{X}\,\exp\left(-\frac{(q-\overline{X})^2}{\lambda^2}\right)\theta_3\left(\frac{\mathrm{i}(q-\overline{X})b}{\lambda^2}\middle|\, 2\tau\right) \tag{22}$$

where τ is given in the line following equation (11). The integrand in equation (22) is periodic in q, which is not seen directly, but becomes evident after performing a Jacobi imaginary transformation [16] on the θ_3-function. We then have for I in equation (22)

$$I = \frac{1}{2\pi}\int_{-b/2}^{b/2}\overline{X}\,\theta_3\left(\frac{\pi(q-\overline{X})}{b}\middle|\,\mathrm{i}\,\frac{\pi\,\lambda^2}{b^2}\right)\mathrm{d}\overline{X} \tag{23}$$

with the integrand now demonstratively periodic in q and we obtain the following final result

$$\overline{X}_{mn,m'n'} = -\mathrm{i}\,\delta_{mm'}\,\frac{b}{2\pi}\frac{(-1)^{(m+1)(n'-n)}}{n'-n}\,\exp\left[-\frac{\pi^2\,\lambda^2}{b^2}\,(n'-n)^2\right]. \tag{24}$$

In exactly the same way we find for the momentum

$$\overline{P}_{mn,m'n'} = \mathrm{i}\,\delta_{nn'}\,\frac{\hbar}{b}\frac{(-1)^{(n+1)(m'-m)}}{m'-m}\,\exp\left[-\frac{b^2}{4\lambda^2}\,(m'-m)^2\right]. \tag{25}$$

Before interpreting the results in equations (24) and (25), let us first calculate the uncertainties Δx and Δp. We already have the expectation values $\overline{x}$ and $\overline{p}$ (see equation (20)). We then need to calculate $\overline{(x^2)}$ and $\overline{(p^2)}$. For this we have to calculate the integral

$$\frac{1}{h}\iint\langle 0|D(-\beta)\,D(-\alpha_{mn})|\,L^2|D(\alpha_{mn}+\beta)|0\rangle\,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} \tag{26}$$

where now L is either x or p. The calculation of the uncertainties is entirely elementary (we just use the expressions of x and p via the annihilation and creation operators) and the results are

$$(\Delta x)^2 = \lambda^2\left[\frac{1}{12}\left(\frac{b}{\lambda}\right)^2 + \frac{1}{2}\right] \qquad (\Delta p) = \frac{\hbar^2}{\lambda^2}\left[\frac{4\pi^2}{12}\left(\frac{\lambda}{b}\right)^2 + \frac{1}{2}\right]. \tag{27}$$

As is easily seen, the product of the uncertainties depends on the ratio $(\frac{b}{\lambda})$ only. Assuming that λ is fixed for the ground state of the harmonic oscillator, we look for the value of the constant b at which $\Delta x\, \Delta p$ becomes a minimum, and the value $(\Delta x\, \Delta p)_{\min}$ of the minimum. The result is

$$b = \lambda\sqrt{2\pi} \qquad \text{and} \qquad (\Delta x\,\Delta p)_{\min} = \frac{2\pi+6}{12}\,\hbar \approx \hbar. \tag{28}$$

This result is very interesting. It shows that the minimum is achieved for $b = \lambda\sqrt{2\pi}$, which is the case of a square lattice [7]. For this value of b the spread of $\Delta x\, \Delta p$ is very close to $\frac{h}{2\pi}$, which is by the factor 2π smaller than the area of the unit cell h. This means that our averaging procedure should enable one to distinguish experimentally between unit cells in the phase plane, and thus measure simultaneously the co-ordinate (mb) and the momentum $(n \frac{2\pi}{b} \hbar)$ of a single unit cell. A verification for this is also obtained from the expressions for the matrix elements for x and p (equations (16), (17), (24) and (25)). For $b = \lambda \sqrt{2\pi}$ the exponent in the off-diagonal elements is the same in both equations and equals $\exp(-\frac{\pi}{2})$. This means that the largest off-diagonal elements (for $n' - n = \pm 1$ and $m' - m = \pm 1$) are by a factor of $\frac{\exp(-\frac{\pi}{2})}{2\pi}$ (close to 20) smaller than the diagonal ones. One can therefore consider in the lowest approximation the matrices for x and p (equations (16) and (17)) as diagonal with mb and $n \frac{2\pi}{b} \hbar$ respectively on the diagonals.

We would like to point out that the averaging procedure developed here differs from the ideas commonly used in the literature [17, 18] on simultaneous measurements of co-ordinates and momentum. While our procedure is a direct arithmetical averaging on a unit cell of the von Neumann lattice, the previously used approach to the problem [17, 18] is usually based on phase space distributions, like the Wigner [19] or the Husimi [11] distribution functions.

The unit cell averaging procedure can also be used for defining a probability distribution on the von Neumann lattice for any state $|\psi\rangle$:

$$|\psi(m,n)|^2 = \frac{1}{h} \iint |\langle 0|D(-\beta)\, D(-\alpha_{mn})|\psi\rangle|^2 \,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P}. \tag{29}$$

This definition is nothing else but the Husimi function [11] averaged on the unit cell of a von Neumann lattice. Having orthogonalized the coherent states, the quantity $|\psi(m,n)|^2$ is now the probability for a system described by $|\psi\rangle$ to be found in the $(mb, n\frac{2\pi}{b}\hbar)$-unit cell on the phase plane. Calculations of $|\psi(m,n)|^2$ are straightforward because $\langle 0|D(-\beta)\, D(-\alpha_{mn})|\psi\rangle$ is directly related to the Bargmann representation [20], on which there has been much information [2–5], and we will not deal with them here. Instead, let us write the scalar product entering equation (29) in the kq-representation

$$(-1)^{mn}\langle 0|D(-\beta)\, D(-\alpha_{mn})|\psi\rangle = \int [D(\beta)\langle kq|0\rangle]^* C(k,q) \exp\left(\mathrm{i}kbm - \mathrm{i}q\frac{2\pi}{b}n\right) \mathrm{d}k\,\mathrm{d}q \tag{30}$$

where $C(k,q)$ is the kq-transform of $|\psi\rangle$. In signal processing the quantity in equation (30) is called a windowed Fourier transform [21]. By inverting equation (30) we have

$$[D(\beta)\langle kq|0\rangle]^* C(k,q) = \frac{1}{2\pi} \sum_{m,n} (-1)^{mn} \langle 0|D(-\beta)\, D(-\alpha_{mn})\, \psi\rangle \exp\left(-\mathrm{i}kbm + \mathrm{i}q\frac{2\pi}{b}n\right) \tag{31}$$

which is a Fourier expansion of the periodic function $[D(\beta)\langle kq|0\rangle]^* C(k,q)$ in k with period $\frac{2\pi}{a}$ and in q with period a. It follows

$$\sum_{mn} |\langle 0|D(-\beta)\, D(-\alpha_{mn})|\psi\rangle|^2 = 2\pi \int |D(\beta)\langle kq|0\rangle|^2\, |C(k,q)|^2\, \mathrm{d}k\,\mathrm{d}q. \tag{32}$$

Averaging both sides of equation (32) over the unit cell in phase plane gives (see equation (29))

$$\sum_{mn} |\psi(m,n)|^2 = \frac{1}{h} \iint \sum_{mn} |\langle 0|D(-\beta)\, D(-\alpha_{mn})|\psi\rangle|^2 \,\mathrm{d}\overline{X}\,\mathrm{d}\overline{P} = 1. \tag{33}$$

With the sum of the squares summing up to 1, this is another indication that $|\psi(m,n)|^2$ is a probability distribution over the discrete set of unit cells in the phase plane.

The expansion in equation (31) can be used for finding the function $C(k,q)$ (if the expansion coefficients $\langle 0|D(-\beta)\, D(-\alpha_{mn})\, \psi\rangle$ are known) by dividing both sides of equation (31) by $[D(\beta)\langle kq|0\rangle]^*$. A difficulty arises because $D(\beta)\langle kq|0\rangle$ (like every continuous

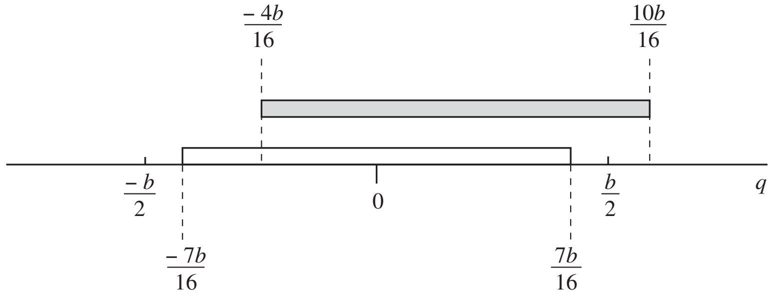

Figure 1. Intervals on the q-axis where $\langle k, q|0\rangle$ (open bar) and the shifted one $\langle k, q - \frac{3b}{16}|0\rangle$ (shaded bar) have no zeros for any value of k. These intervals cover the range of $\frac{17b}{16}$ which exceeds the period b of the q co-ordinate.

kq-function) has a zero at some point in the unit cell [7, 22, 23]. Thus, $\langle kq|0\rangle$ has a zero at $k = \frac{\pi}{b}, q = \frac{b}{2}$. But in the expansion in equation (31) we have the shift operator $D(\beta)$ that can be used as a tool for shifting the zero. Thus, for $\beta = \frac{1}{\lambda\sqrt{2}}\frac{3b}{16}$, the function $D(\frac{1}{\lambda\sqrt{2}}\frac{3b}{16})\langle kq|0\rangle$ has a zero at $(\frac{\pi}{b}, \frac{11b}{16})$. With the shift operator in our disposal, one can avoid the zero by carrying out the expansion in equation (31) in two overlapping intervals on the q axis and for any k

$$\begin{aligned} &\text{Interval (1)} \quad -\frac{7b}{16} \leqslant q \leqslant \frac{7b}{16} \\ &\text{Interval (2)} \quad -\frac{4b}{16} \leqslant q \leqslant \frac{10b}{16}. \end{aligned} \tag{34}$$

These intervals are plotted in figure 1, where interval (1) is for $\langle kq|0\rangle$ and interval (2) for $\langle k, q - \frac{3b}{16}|0\rangle$. Equation (31) can therefore serve for finding the function $C(k, q)$ from the coefficients $\langle 0|D(\beta)\, D(\alpha_{mn})\,\psi\rangle$ for the two intervals with $\beta = 0$ and $\frac{1}{\lambda\sqrt{2}}\frac{3b}{16}$ (see equations (1) and (9) and figure 1).

In summary, we have developed an averaging procedure making the coherent states on a von Neumann lattice orthonormal.

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